

Macroscopic Superpositions, Decoherent Histories and the Emergence of Hydrodynamic Behaviour

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Abstract

Macroscopic systems are described most completely by local densities (particle number, momentum and energy) yet the superposition states of such physical variables, indicated by the Everett interpretation, are not observed. In order to explain this, it is argued that histories of local number, momentum and energy density are approximately decoherent when coarse-grained over sufficiently large volumes. Decoherence arises directly from the proximity of these variables to exactly conserved quantities (which are exactly decoherent), and not from environmentally-induced decoherence. We discuss the approach to local equilibrium and the subsequent emergence of hydrodynamic equations for the local densities. The results are general but we focus on a chain of oscillators as a specific example in which explicit calculations may be carried out. We discuss the relationships between environmentally-induced and conservation-induced decoherence and present a unified view of these two mechanisms.

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I. INTRODUCTION

If the Everett interpretation of quantum theory is to be taken seriously, there will exist superposition states for macroscopic systems, perhaps even for the entire universe. Since such states are not observed, it is therefore necessary to explain why they go away. This question is a key part of the general question of the emergence of classical behaviour from quantum theory, an issue that has received a considerable amount of attention [1, 2].

There are a number of different approaches to emergent classicality, but common to most of them is the notion that there must be decoherence, that is, that certain types of quantum states of the system in question exhibit negligible interference, and therefore superpositions of them are effectively equivalent to statistical mixtures. Decoherence has been extensively investigated for the situation in which there is a distinguished system, such as a particle, coupled to its surrounding environment [3, 4]. However, for many macroscopic systems, and in particular for the universe as a whole, there may be no natural split into distinguished subsystems and the rest and another way of identifying the naturally decoherent variables is required. Most generally, decoherence comes about when the variables describing the entire system of interest naturally separate into “slow” and “fast”, whether or not this separation corresponds to, respectively, system and environment. If the system consists of a large collection of interacting identical particles, such as a fluid for example, the natural set of slow variables are the local densities: energy, momentum, number, charge *etc.* They are “slow” because they are locally conserved. These variables, in fact, are also the variables which provide the most complete description of the classical state of a fluid at a macroscopic level. The most general demonstration of emergent classicality therefore consists of showing that, for a large collection of interacting particles described microscopically by quantum theory, the local densities become effectively classical. Although decoherence through the system–environment mechanism may play a role, since the collection of particles are coupled to each other, it is important to explore the possibility that, at least in some regimes, decoherence could come about because the local densities are almost conserved if averaged over a sufficiently large volume [5]. Hence, the approximate decoherence of local densities would then be due to their proximity to a set of exactly conserved quantities, and exactly conserved quantities obey superselection rules.

We will approach these questions using the decoherent histories approach to quantum the-

ory [5–10]. This approach has proved particularly useful for discussing emergent classicality in a variety of contexts. In particular the issues outlined above are most clearly expressed in the language of decoherent histories. The central object of interest is the decoherence functional,

$$D(\underline{\alpha}, \underline{\alpha}') = \text{Tr} (P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1) \rho P_{\alpha'_1}(t_1) \cdots P_{\alpha'_n}(t_n)) \quad (1.1)$$

The histories are characterized by the initial state ρ and by the strings of projection operators $P_\alpha(t)$ (in the Heisenberg picture) at times t_1 to t_n (and $\underline{\alpha}$ denotes the string of alternatives $\alpha_1 \cdots \alpha_n$). Intuitively, the decoherence functional is a measure of the interference between pairs of histories $\underline{\alpha}, \underline{\alpha}'$. When it is zero for $\underline{\alpha} \neq \underline{\alpha}'$, we say that the histories are decoherent and probabilities $p(\underline{\alpha}) = D(\underline{\alpha}, \underline{\alpha})$ obeying the usual probability sum rules may be assigned to them. One can then ask whether these probabilities are strongly peaked about trajectories obeying classical equations of motion. For the local densities, we expect that these equations will be hydrodynamic equations.

The aim of this paper is review this programme, following primarily Refs.[11–14]. We will outline the argument showing how the approximate conservation of the local densities implies negligible interference of their histories at sufficiently coarse grained scales, and show how hydrodynamic equations of motion for them arise.

II. LOCAL DENSITIES AND HYDRODYNAMIC EQUATIONS

We are generally concerned with a system of N particles described at the microscopic level by a Hamiltonian of the form

$$H = \sum_j \frac{\mathbf{p}_j^2}{2m} + \sum_{\ell > j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell) \quad (2.1)$$

We are particularly interested in the number density $n(\mathbf{x})$, the momentum density $\mathbf{g}(\mathbf{x})$ and the energy density $h(\mathbf{x})$, defined by,

$$n(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{q}_j) \quad (2.2)$$

$$\mathbf{g}(\mathbf{x}) = \sum_j \mathbf{p}_j \delta(\mathbf{x} - \mathbf{q}_j) \quad (2.3)$$

$$h(\mathbf{x}) = \sum_j \frac{\mathbf{p}_j^2}{2m} \delta(\mathbf{x} - \mathbf{q}_j) + \sum_{\ell > j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell) \delta(\mathbf{x} - \mathbf{q}_j) \quad (2.4)$$

We are interested in the integrals of these quantities over volumes which are large compared to the microscopic scale but small compared to macroscopic physics. Integrated over an infinite volume, these become the total particle number N , total momentum P and total energy H , which are exactly conserved. It is also often more useful to work with the Fourier transforms of the local densities,

$$n(\mathbf{k}) = \sum_j e^{i\mathbf{k}\cdot\mathbf{q}_j} \quad (2.5)$$

$$\mathbf{g}(\mathbf{k}) = \sum_j \mathbf{p}_j e^{i\mathbf{k}\cdot\mathbf{q}_j} \quad (2.6)$$

$$h(\mathbf{k}) = \sum_j \frac{\mathbf{p}_j^2}{2m} e^{i\mathbf{k}\cdot\mathbf{q}_j} + \sum_{\ell>j} V_{j\ell}(\mathbf{q}_j - \mathbf{q}_\ell) e^{i\mathbf{k}\cdot\mathbf{q}_j} \quad (2.7)$$

These quantities tend to the exactly conserved quantities in the limit $k = |\mathbf{k}| \rightarrow 0$, so we are interested in what happens in what happens for small but non-zero k .

Setting aside for the moment the issues of decoherence, there is a standard technique for deriving hydrodynamic equations for the local densities [2, 15, 16]. It starts with the continuity equations expressing local conservation, which have the form,

$$\frac{\partial\sigma}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (2.8)$$

where σ denotes n , \mathbf{g} or h (and the current \mathbf{j} is a second rank tensor in the case of \mathbf{g}). It is then assumed that, for a wide variety of initial states, conditions of local equilibrium are established after a short period of time. This means that on scales small compared to the overall size of the fluid, but large compared to the microscopic scale, equilibrium conditions are reached in each local region, characterized by a local temperature, pressure *etc.* which vary slowly in space and time. Local equilibrium is described by the density operator

$$\rho = Z^{-1} \exp \left(- \int d^3x \beta(\mathbf{x}) [h(\mathbf{x}) - \bar{\mu}(\mathbf{x})n(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})] \right) \quad (2.9)$$

where β , $\bar{\mu}$ and \mathbf{v} are Lagrange multipliers and are slowly varying functions of space and time. β is the inverse temperature, \mathbf{v} is the average velocity field, and $\bar{\mu}$ is related to the chemical potential which in turn is related to the average number density. (Note that the local equilibrium state is defined in relation to a particular coarse-graining, here, the anticipated calculation of average values of the local densities. Hence it embraces all possible states that are effectively equivalent to the state Eq.(2.9) for the purposes of calculating those

averages.) The hydrodynamic equations follow when the continuity equations are averaged in this state. These equations form a closed set because the local equilibrium form depends (in three dimensions) only on the five Lagrange multiplier fields $\beta, \bar{\mu}, \mathbf{v}$ and there are exactly five continuity equations (2.8) for them. (More generally, it is possible to have closure up to a set of small terms which may be treated as a stochastic process. See Refs.[17, 18], for example.)

We will in this paper concentrate on the useful pedagogical example of a chain of oscillators, in which many calculations can be carried out explicitly [11]. The Hamiltonian of this system is

$$H = \sum_{n=1}^N \left[\frac{p_n^2}{2m} + \frac{\nu^2}{2} (q_n - q_{n-1})^2 + \frac{K}{2} (q_n - b_n)^2 \right] \quad (2.10)$$

There are two cases $K = 0$ (the simple chain) and $K \neq 0$ (the harmonically bound chain). In the bound chain case, it is also useful to consider the case $b_n = 0$, which corresponds to the situation in which the whole chain moves in a harmonic potential. We consider a finite number N of particles but it is sometimes useful to approximate N as infinite.

The local densities of this system are

$$n(x) = \sum_{n=1}^N \delta(q_n - x) \quad (2.11)$$

$$g(x) = \sum_{n=1}^N p_n \delta(q_n - x) \quad (2.12)$$

$$h(x) = \sum_{n=1}^N \left[\frac{p_n^2}{2m} + \frac{\nu^2}{2} (q_n - q_{n-1})^2 + \frac{1}{2} K (q_n - b_n)^2 \right] \delta(q_n - x) \quad (2.13)$$

They satisfy the local conservation laws

$$\dot{n}(x) = -\frac{1}{m} \frac{\partial g}{\partial x} \quad (2.14)$$

$$\dot{g}(x) = -\frac{\partial \tau}{\partial x} - K x n(x) + K \sum_j b_j \delta(q_j - x) \quad (2.15)$$

$$\dot{h}(x) = -\frac{\partial j}{\partial x} \quad (2.16)$$

The currents $\tau(x)$ and $j(x)$ are rather complicated in configuration space, except in the case

where we neglect the interaction term, when they are given by

$$\tau(x) = \sum_j \frac{p_j^2}{m} \delta(q_j - x) \quad (2.17)$$

$$j(x) = \sum_j \frac{p_j}{m} \left(\frac{p_j^2}{2m} + \frac{1}{2} K (q_j - b_j)^2 \right) \delta(q_j - x) \quad (2.18)$$

The standard derivation of the hydrodynamic equations may be carried out reasonably easily in this model. Instead of the density operator form Eq.(2.9) of the local equilibrium state, we work with the equivalent one-particle Wigner function (phase space density)

$$w_j(p_j, q_j) = f(q_j) \exp \left(-\frac{(p_j - mv(q_j))^2}{2mkT(q_j)} \right) \quad (2.19)$$

where f , v and T are slowly varying functions of space and time (f is simply related to the chemical potential in Eq.(2.9)). This is the one-particle distribution function for particle j – it is labelled by j since the particles are distinguishable. If we now average the system Eqs.(2.14)–(2.16), together with the currents $\tau(x)$, $j(x)$ in the local equilibrium state, we obtain a closed system, since we get three equations for three unknowns. In the case of negligible interactions and $b_j = 0$, we find

$$\langle n(x) \rangle = Nf(x) \quad (2.20)$$

$$\langle g(x) \rangle = mv(x)Nf(x) \quad (2.21)$$

$$\langle h(x) \rangle = \left(\frac{1}{2}mv^2 + \frac{1}{2}kT + \frac{1}{2}Kx^2 \right) Nf(x) \quad (2.22)$$

$$\langle \tau(x) \rangle = (mv^2 + kT) Nf(x) \quad (2.23)$$

$$\langle j(x) \rangle = \left(\frac{3}{2}vkT + \frac{1}{2}mv^3 \right) Nf(x) + \frac{K}{2m}x^2 \langle g(x) \rangle \quad (2.24)$$

The first three equations give the explicit inversion between the averages of the local densities and the three slowly varying functions f, v, T . Inserted in Eqs.(2.14)–(2.16), the above relations give a closed set of equations for the three variables f , v and T . After some rearrangement, these equations are

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = -f \frac{\partial v}{\partial x} \quad (2.25)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{m} \frac{\partial \theta}{\partial x} - \frac{\theta}{mf} \frac{\partial f}{\partial x} - \frac{Kx}{m} \quad (2.26)$$

$$\frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{\partial x} = -2\theta \frac{\partial v}{\partial x} \quad (2.27)$$

where $\theta = kT$. These are the equations for a one-dimensional fluid moving in a harmonic potential [15]. Note that non-trivial equations are obtained even though we have neglected the interaction terms in deriving them. The role of interactions is to ensure the approach to local equilibrium, as we discuss below.

In these expressions, the definition of the temperature fields is essentially equivalent to,

$$\sum_j \frac{1}{2m} (\Delta p_j)^2 \delta(q_j - x) = \frac{1}{2} kT(x) n(x) \quad (2.28)$$

(recalling that we are working at long wavelengths, so the δ -function is coarse-grained over a scale of order k^{-1}). Hence temperature arises not from an environment, but from the momentum fluctuations averaged over a coarse-graining volume.

It is straightforward to give a decoherent histories version of the standard derivation of the hydrodynamic equations. We take the initial state to be the local equilibrium state. We take the histories to be characterized by projection operators onto broad ranges of values of the local densities. (The local densities do not commute in general, but they will approximately commute for sufficiently small k and it is not difficult to construct quasi-projectors that are well-localized in all three densities). Then, it is easily shown that for sufficiently broad projections, the histories are peaked about the average values of the local densities, averaged in the initial local equilibrium state [13]. The standard derivation shows that the average values obey hydrodynamic equations hence the probabilities are peaked about evolution according to those equations.

However, what is important here is that the decoherent histories approach to quantum theory offers the possibility of a derivation of emergent classicality much more general than that entailed in the standard derivation of hydrodynamics. The standard derivation is rather akin to the Ehrenfest theorem of elementary quantum mechanics which shows that the averages of position and momentum operators obey classical equations of motion. Yet a description of emergent classicality must involve much more than that [1]. Firstly, it must demonstrate decoherence of the local densities, thereby allowing us to talk about probabilities for their histories. Secondly, it should not be restricted to a special initial state. Whilst it is certainly plausible that many initial states will tend to the local equilibrium state, the standard derivation does not obviously apply to superpositions of macroscopic states, which are exactly the states a description of emergent classicality is supposed to deal with. It is to this more general derivation that we now turn.

III. DECOHERENCE AND CONSERVATION

We begin by describing the connection between decoherence and conservation. It is well-known that histories of exactly conserved quantities are exactly decoherent [19]. The simple reason for this is that the projectors onto conserved quantities commute with the Hamiltonian. The projectors P_{α_k} on one side of the decoherence functional (1.1) may therefore be brought up against the projectors $P_{\alpha'_k}$ on the other side, hence the decoherence functional is exactly diagonal. (In the situation considered here, in which there are three conserved quantities involved, these quantities must in addition commute with each other, but this is clearly the case.)

There is another way of expressing this that is more useful for the generalization to the local densities. Suppose we take the initial state to be a pure state $|E, \mathbf{P}, N\rangle$ which is an eigenstate of the total energy, momentum and number, and consider a history of projections onto total energy, momentum and number. Clearly, unitary evolution preserves the eigenstate (except for a phase), and the projections acting on it either give back the state, or give zero. This means that

$$P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1)|E, \mathbf{P}, N\rangle \quad (3.1)$$

is equal to either $|E, \mathbf{P}, N\rangle$ or to zero. It is easy to see, by expanding an arbitrary initial state in eigenstates of the conserved quantities that this implies exact decoherence of histories for any initial state.

Turning now to the local densities, which are most usefully discussed in the Fourier transformed form Eqs.(2.5)-(2.7), the above argument shows that they define exactly decoherent sets of histories for the case $k = 0$. Now here is the key point: as k departs from zero, the decoherence functional will depart from exact diagonality, but there will still clearly be approximate decoherence if k is sufficiently small. That is, decoherence of local densities essentially follows from an expansion for small k about the exactly decoherent case, $k = 0$. The aim of much of the rest of this section is to spell out in more detail how this works out.

We generalize the above argument for exact decoherence of histories of conserved quantities, to locally conserved quantities. We suppose we have a set of histories characterized by projections onto the local densities for some value of k . We then consider states $|h, \mathbf{g}, n\rangle$ which are approximate eigenstates of the local densities. Exact eigenstates are not possible,

but it is not hard to find states which are well-localized all three variables. Under time evolution, the local density eigenstates $|h, \mathbf{g}, n\rangle$ will not remain exact eigenstates, but as long as they remain approximate eigenstates (that is, well-localized in the local densities), the above argument goes through and we deduce approximate decoherence.

Hence, denoting the local densities by Q , what we need to show is that, for an initial state localized in the local densities, under time evolution, Q satisfies the condition,

$$\frac{(\Delta Q(t))^2}{\langle Q(t) \rangle^2} \ll 1 \quad (3.2)$$

where

$$(\Delta Q(t))^2 = \langle Q^2(t) \rangle - \langle Q(t) \rangle^2 \quad (3.3)$$

Eq.(3.2) means that the state remains strongly peaked in the variable Q under time evolution. The states are then approximate eigenstates of the projectors at each time as long as the widths of the projectors are chosen to be much greater than $(\Delta Q(t))^2$. The condition Eq.(3.2) must be true approximately for some $k \neq 0$ since it holds exactly in the limit $k \rightarrow 0$. The question is to determine the lengthscale involved.

The number and momentum density are both operators of the form,

$$A = \sum_{n=1}^N A_n \quad (3.4)$$

as is the local energy density, if we ignore the interaction term. For such operators it follows that

$$(\Delta A)^2 = \sum_n (\Delta A_n)^2 + \sum_{n \neq m} \sigma(A_n, A_m) \quad (3.5)$$

and

$$\langle A \rangle^2 = \sum_{n,m} \langle A_n \rangle \langle A_m \rangle \quad (3.6)$$

where the correlation function σ is defined by

$$\sigma(A, B) = \frac{1}{2} \langle AB + BA \rangle - \langle A \rangle \langle B \rangle \quad (3.7)$$

A state will be an approximate eigenstate of the operator A if

$$\frac{(\Delta A)^2}{\langle A \rangle^2} \ll 1 \quad (3.8)$$

The expression for $\langle A \rangle^2$ potentially involves N^2 terms, as does the expression for $(\Delta A)^2$, but the latter will involve only N terms if the correlation functions $\sigma(A_n, A_m)$ are very small

or zero for $n \neq m$. So simple product states will be approximate eigenstates and will have $(\Delta A)^2/\langle A \rangle^2$ of order $1/N$. (See Refs.[12, 13] for more detailed examples this argument).

Under time evolution, the interactions cause correlations to develop. However, the states will remain approximate eigenstates as long as the correlations are sufficiently small that the second term in Eq.(3.5) is much smaller than order N^2 . The interactions and the subsequent correlations are clearly necessary in order to get interesting dynamics and in particular the approach to local equilibrium. The interesting question is therefore whether there is a regime where the effects of interactions are small enough to permit decoherence but large enough to produce interesting dynamics. The fact that the variables we are interested in are locally conserved indicates that there is such a regime. The important point is that the local densities become arbitrarily close to exactly conserved quantities as $k \rightarrow 0$. This means that, at any time, $(\Delta A)^2/\langle A \rangle^2$ becomes arbitrarily close to its initial value (which is of order $1/N$) for sufficiently small k .

In specific examples an uncorrelated initial state develops correlations with a typical lengthscale. These correlations typically then decay with time. What is found is that the second term in Eq.(3.5) will remain small as long as k^{-1} is much greater than the correlation length. Hence the key physical aspect is the locality of the interactions, meaning that only limited local correlations develop, together with the coarse-graining scale k^{-1} which may be chosen to be sufficiently large that the correlation scale is not seen. Differently put, as k increases from zero, departing from exact decoherence, it introduces a lengthscale k^{-1} . Since the decoherence functional is a dimensionless quantity, clearly nothing significant can happen until k^{-1} becomes comparable with another lengthscale in the system. The natural scale is the correlation length in the local density eigenstates.

IV. CHAINS OF OSCILLATORS

Some of the claims of the physical ideas of the previous section may be seen explicitly in the oscillator model with Hamiltonian Eq.(2.10). The equations of motion are

$$m\ddot{q}_n + K(q_n - b_n) = \nu^2(q_{n+1} - 2q_n + q_{n-1}) \quad (4.1)$$

where we take $q_{N+1} = q_1$. This system has been discussed and solved in many places [20–25]. The solution may be written,

$$q_n(t) = b_n + \sum_{r=1}^N \left[f_{r-n}(t) q_r(0) + \frac{g_{r-n}(t)}{m\Omega} p_r(0) \right] \quad (4.2)$$

where, $\Omega^2 = (K + 2\nu^2)/m$. For the bound chain, $K \neq 0$, it is most useful to work in the regime in which the interaction between particles is much weaker than the binding to their origins, so $\nu^2 \ll K$. In this case, the functions $f_r(t)$ and $g_r(t)$ are given by [20],

$$f_r(t) \approx J_r(\gamma\Omega t) \cos(\Omega t - \pi r/2) \quad (4.3)$$

and

$$g_r(t) \approx J_r(\gamma\Omega t) \sin(\Omega t - \pi r/2) \quad (4.4)$$

where $\gamma = \nu^2/m\Omega^2$, so $\gamma \ll 1$ and J_r is the Bessel function of order r (and we have used the convenient approximation of taking N to be infinite).

The general behaviour of the solutions is easily seen. The functions $f_{r-n}(t)$ and $g_{r-n}(t)$ loosely represent the manner in which an initial disturbance of particle r affects particle n after a time t , and is given by the properties of Bessel functions [26]. $J_n(x)$ decays rapidly for large n at fixed x , so distant particles do not affect each other very much. Evolving in x , $J_n(x)$ starts at zero for $x = 0$ (except for $n = 0$, where $J_0(0) = 1$), grows exponentially, and then goes into a slowly decaying oscillation,

$$J_n(x) \sim \left(\frac{2}{\pi x} \right)^{1/2} \cos(x - \pi n/2 - \pi/4) \quad (4.5)$$

In this oscillatory regime, the Bessel function $J_n(x)$ has only a very limited dependence on n , namely it has the form (4.5) for some n , plus the three possible phase shifts of $\pi/2$. This means that conditions along the chain do not vary very much for reasonably large sections, which relates to the establishment of local equilibrium.

These classical solutions may be used to determine the time evolution of the correlation functions such as $\sigma(q_n, q_m)$, $\sigma(q_n, p_m)$ and $\sigma(p_n, p_m)$ which are the key quantities determining the behaviour of the local densities under time evolution. In brief, what is found is the following. An initially uncorrelated state will develop correlations, but these then decay with time, with the correlations never becoming too great. Furthermore, the quantities $(\Delta q_n)^2$ and $(\Delta p_n)^2$ become dependent only very weakly on n , indicating a situation similar to local equilibrium.

Now consider the local densities of the oscillator chain. For simplicity, we focus on the number density $n(k)$, given by the one-dimensional version Eq.(2.5). Following the general scheme outlined in the previous section, we consider initial states which are approximate eigenstates of the local densities. Gaussian states suffice, in fact, and these will be approximate eigenstates of the local densities if we choose the correlation functions $\sigma(q_n, q_m)$, $\sigma(q_n, p_m)$ and $\sigma(p_n, p_m)$ to be zero, or at least sufficiently small, for $n \neq m$.

In a general Gaussian state, we have

$$\langle n(k) \rangle = \sum_{j=1}^N \langle e^{ikq_j} \rangle = \sum_{j=1}^N \exp \left(ik \langle q_j \rangle - \frac{1}{2} k^2 (\Delta q_j)^2 \right) \quad (4.6)$$

and

$$\begin{aligned} (\Delta n(k))^2 &= \langle n^\dagger(k) n(k) \rangle - |\langle n(k) \rangle|^2 \\ &= \sum_{j=1}^N \sum_{n=1}^N \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle \left(e^{k^2 \sigma(q_j, q_n)} - 1 \right) \end{aligned} \quad (4.7)$$

The latter is to be compared with

$$|\langle n(k) \rangle|^2 = \sum_{j=1}^N \sum_{n=1}^N \langle e^{ikq_j} \rangle \langle e^{-ikq_n} \rangle \quad (4.8)$$

With an initially uncorrelated state we have $\sigma(q_j, q_n) = 0$ for $j \neq n$ and we see that

$$(\Delta n(k))^2 = \sum_j |\langle e^{ikq_j} \rangle|^2 \left(e^{k^2 (\Delta q_j)^2} - 1 \right) \quad (4.9)$$

From this we expect that

$$(\Delta n(k))^2 \ll |\langle n(k) \rangle|^2 \quad (4.10)$$

as long as k^{-1} does not probe on scales that are too short (compared to Δq_j), and in this case the Gaussian state is an approximate eigenstate as required.

Under time evolution, correlations develop, but we expect that the state will remain an approximate eigenstate if k^{-1} is much greater than the lengthscale of correlation. As k increases from zero we have, to leading order in small k ,

$$\frac{(\Delta n(k))^2}{|\langle n(k) \rangle|^2} \sim \frac{k^2 (\Delta X)^2}{N^2} \quad (4.11)$$

where $X = \sum_j q_j$ (the centre of mass coordinate of the whole chain). This will be very small as long as k^{-1} is much larger than the typical lengthscale of a single particle. $(\Delta n(k))^2$ starts

to grow very rapidly with k , and Eq.(4.10) is no longer valid, when k^{-1} becomes less than the correlation length indicated by $\sigma(q_j, q_n)$. Hence the local density eigenstate state remains strongly peaked about the mean as long as the coarse graining lengthscale k^{-1} remains much greater than the correlation length, confirming the general arguments of the previous section. Similarly, it can be argued that the local density eigenstates also remain localized in the local energy and momentum. This shows that there is approximate decoherence of histories in the oscillator chain model, confirming the general argument.

V. THE APPROACH TO LOCAL EQUILIBRIUM

Given decoherence, we may now look at the probabilities for histories and see if they are peaked around interesting evolution equations. Since we have shown that there is negligible interference between histories with an initial state consisting of a superposition of local density eigenstates, we may take the initial state in these probabilities to be a local density eigenstate. Decoherence alone is not enough to get the hydrodynamic equations. Decoherence ensures that the probabilities for histories are well-defined but the probabilities may not be peaked around any particularly interesting histories and in fact will typically describe a situation which is highly stochastic. The hydrodynamic equations we seek form a *closed* set of equations. This requires at least two things in the histories description. First of all, it requires that we consider histories specified by a sufficiently large number of variables – all three of the local densities, particle number, momentum, energy, are required. It is not enough to consider histories of just one of them. Even classically, the momentum density, for example, will generally not obey a closed evolution equation on its own. Hence, we will assume that histories of all three local densities are considered.

Secondly, the hydrodynamic equations emerge only when the initial state is a local equilibrium state. We need to show how this state, a mixed state, arises from the local density eigenstate, a pure state defined very differently. The point here is that for sufficiently coarse grained projections onto the local densities, the object that will determine the probabilities for histories is ρ_1 , the one-particle density operator constructed by tracing the local density eigenstate. This is actually quite similar to the local equilibrium state, since they are both mixed states localized in the local densities. They differ in that ρ_1 may still contain correlations (and in particular have non-zero $\sigma(p, q)$) not contained in the local equilibrium state.

However, since they are so similar, it is physically extremely plausible that ρ_1 will approach the local equilibrium form on short time scales and this has indeed been explicitly verified in the oscillator model of Ref.[11]. It then follows that the probabilities will be peaked about the hydrodynamic equations.

The final picture we have is as follows. We can imagine an initial state for the system which contains superpositions of macroscopically very distinct states. Decoherence of histories indicates that these states may be treated separately and we thus obtain a set of trajectories which may be regarded as exclusive alternatives each occurring with some probability. Those probabilities are peaked about the average values of the local densities. We have argued that each local density eigenstate may then tend to local equilibrium, and a set of hydrodynamic equations for the average values of the local densities then follow. We thus obtain a statistical ensemble of trajectories, each of which obeys hydrodynamic equations. These equations could be very different from one trajectory to the next, having, for example, significantly different values of temperature. In the most general case they could even be in different phases, for example one a gas, one a liquid.

Decoherence requires the coarse-graining scale k^{-1} to be much greater than the correlation length of the local density eigenstates, and the derivation of the hydrodynamic equations requires $k^{-2} \gg (\Delta q)^2$. In brief, the emergence of the classical domain occurs on lengthscales much greater than any of the scales set by the microscopic dynamics.

VI. CONNECTIONS WITH ENVIRONMENTALLY INDUCED DECOHERENCE

As noted in the Introduction, most studies of decoherence and emergent classicality have focused on the situation in which there is an explicit split into system and environment, and there, the decoherence comes about due to the coarse-graining over environmental variables. What is the connection between environmentally-induced decoherence (EID) and conservation-induced decoherence (CID) considered in this paper? Here we consider three different issues.

First, in EID there is the question of the split into system and environment. Here, the guiding principle is conservation. System usually means a “large” particle, and environment a bunch of “small” particles, but in practice the key difference between them is that large particles are slow and the small ones fast, which relates, approximately, to conservation of

something, such as number or momentum density. (Although there is typically no limit of exact conservation).

Second, there is a unified way of seeing decoherence of histories in the two cases. Denote a generic variable by $A(t)$. Decoherence of histories of A follows when $A(t)$ commutes with itself at different times. Commutation and the resultant decoherence is usually not exact, so approximate decoherence follows when a condition something like this holds:

$$\| A(t_2)A(t_1) + A(t_1)A(t_2) \| \gg \| [A(t_2), A(t_1)] \| \quad (6.1)$$

That is, the anticommutator is much bigger than the commutator in some suitably defined operator norm $\| \cdots \|$.

For CID, A is one or more of the hydrodynamic variables $n(k)$, $g(k)$, $h(k)$. These quantities are exactly conserved at $k = 0$ so commute with their values at different times. The inequality Eq.(6.1) can be satisfied because the right hand side of this inequality may be made arbitrarily small by taking k sufficiently small.

For EID, A is typically the position x of a Brownian particle coupled to an environment and $x(t)$ denotes evolution with the total (system plus environment) Hamiltonian. The norm includes a trace over the environment in a thermal state. The right hand side will be proportional to $1/M$ (M is the mass of the particle) which will be "small" due to the massiveness (slowness) of the particle and it will also be proportional to \hbar . However, what is more important is that, because of the thermal fluctuations, the left hand side will be large - it is typically of order $(\Delta x)^2$ which grows with time and with temperature of the environment. This corresponds to the known fact that EID comes about when thermal fluctuations are much larger than quantum ones.

In brief, Eq.(6.1) gives a unified picture of decoherence of histories. It is satisfied in CID because the right hand side can be made small and in EID because the left hand side can be made large.

A third issue is the question of the relative roles of CID and EID in a given system, since one might generally expect that both mechanisms will operate. The point is that it is a question of lengthscales. We have demonstrated decoherence of the local densities starting with exact conservation at the largest lengthscales and then moving inwards. In this way we were able to prove decoherence without using an environment, for certain sets of histories at very coarse-grained scales whose probabilities are peaked about classical

paths. However, in general we would like to be able to assign probabilities to non-classical trajectories. For example, what is the probability that a system will follow an approximately classical trajectory at a series of times, but then at one particular time undergoes a very large fluctuation away from the classical trajectory? The approach adopted here, based on conservation, would yield an approximately zero probability for this history, to the level of approximation used. Yet this is a valid question that we can test experimentally. It is at this stage that an environment becomes necessary to obtain decoherence, and indeed it is frequently seen in particular models that when there is decoherence of histories due to an environment, decoherence is obtained for a very wide variety of histories, not just histories close to classical. It is essentially a question of information. Decoherence of histories means that information about the histories of the system is stored somewhere [5, 27]. Classical histories need considerably less information to specify than non-classical ones, and indeed specification of the three local densities at any time is sufficient to specify their entire classical histories. This is not enough for non-classical histories, so an environment is required to store the information.

Related to this is the issue of timescales involved in the models considered. Decoherence through interaction with an environment involves a timescale, which is typically exceptionally short. Here, however, there is no timescale associated with decoherence by approximate conservation. The eigenstates of the local densities remain approximate eigenstates for all time. There is, however, a timescale involved in obtaining the hydrodynamic equations, namely, the time required for a local density eigenstate to approach local equilibrium. In this model, this timescale is of order $(\gamma\Omega)^{-1}$ (for the infinite chain in the $K \neq 0$ case).

Finally, and somewhat straying from the issue of EID, we briefly comment on relations to the Boltzmann equation in these models. It would also be of particular interest to look at CID models involving a gas. Many-body field theory may be the appropriate medium in which to investigate this, following the lead of Ref.[28]. The decoherent histories analysis might confer some interesting advantages over conventional treatments. For example, one-particle dynamics of a gas is described by a Boltzmann equation. One of the assumptions involved in the derivation of the Boltzmann equation is that the initial state of the system contains no correlations, which is clearly very restrictive [15]. However, in the general approach used here it is natural to break up any arbitrary initial state into a superposition of local density eigenstates, and that these may then be treated separately because of de-

coherence. The local density eigenstates typically have small or zero correlations. Hence, decoherence gives some justification for one of the rather restrictive assumptions of the Boltzmann equation.

VII. SUMMARY AND DISCUSSION

Physics would be impossible without conservation laws. They are respected by both classical and quantum mechanics and are the key to understanding the emergence of classical behaviour from an underlying quantum theory. We have outlined the process whereby local densities become effectively classical, using local conservation as the guiding principle. The key idea is to split the initial state into local density eigenstates and show that they are preserved in form under time evolution. The subsequent probabilities for histories are peaked about the average values of the local densities, and the equations of motion for them form a closed set of hydrodynamic form on sufficiently large scales, provided, in general, that sufficient time has elapsed for the local density eigenstates to settle down to local equilibrium.

Since this account of emergent classicality is so firmly anchored in conservation laws, and since conservation laws are so central to physics, it seems likely that this account is very general, and will apply to a wide variety of Hamiltonians and initial states: as long as there are conserved quantities there is a regime nearby of almost conserved quantities behaving quasiclassically.

An important general question is whether the quasiclassical domain derived in this way is unique. The familiar quasiclassical domain is characterized by local densities obeying closed sets of deterministic evolution equations. This domain may also be referred to as a *reduced description* of the quantum system, in which, at sufficiently coarse grained scales, certain predictions are possible using only a limited set of variables, the local densities, without having to solve the full quantum theory. Could there be an utterly different domain, characterized by completely different variables, but still obeying deterministic evolution equations? That is, is there another completely different reduced description of the system? The derivation described here rests entirely on conservation laws, and from that point of view, the existence of another quasiclassical domain seems most unlikely, unless there are conservation laws that we have not yet discovered. So perhaps the appropriate question is to ask whether different reduced descriptions of the system are possible that do not rely on

conservation laws. Little is known about this issue at present.

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